

Kloosterman Sums with Multiplicative Coefficients

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Abstract. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, q ($\leq N^2$) be a positive integer and a be an integer with $(a, q) = 1$. In this paper, we shall prove that

$$\sum_{\substack{n \leq N \\ (n, q) = 1}} f(n) e\left(\frac{a\bar{n}}{q}\right) \ll \sqrt{\frac{\tau(q)}{q}} N \log \log(6N) + q^{\frac{1}{4} + \frac{\varepsilon}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}},$$

where \bar{n} is the multiplicative inverse of n such that $\bar{n}n \equiv 1 \pmod{q}$, $e(x) = \exp(2\pi i x)$, $\tau(q)$ is the divisor function.

1. Introduction

Let $\mu(n)$ be the Möbius function, q be a positive integer and a be an integer with $(a, q) = 1$. In 1988, D. Hajela, A. Pollington and B. Smith [8] proved that

$$\sum_{\substack{n \leq N \\ (n, q) = 1}} \mu(n) e\left(\frac{a\bar{n}}{q}\right) \ll_{\varepsilon} N q^{\varepsilon} \left(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{3}{10}} (\log N)^{\frac{11}{5}}}{N^{\frac{1}{5}}} \right),$$

where \bar{n} is the multiplicative inverse of n such that $\bar{n}n \equiv 1 \pmod{q}$, $e(x) = \exp(2\pi i x)$ and ε is a sufficiently small positive constant. This estimate is nontrivial for $(\log N)^{5+10\varepsilon} \ll q \ll N^{\frac{2}{3}-3\varepsilon}$.

Later, P. Deng [4], G. Wang and Z. Zheng [9] independently improved the above estimate to

$$\sum_{\substack{n \leq N \\ (n, q) = 1}} \mu(n) e\left(\frac{a\bar{n}}{q}\right) \ll N \tau(q) \left(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{1}{5}} (\log N)^{\frac{13}{5}}}{N^{\frac{1}{5}}} \right),$$

where $\tau(q)$ is the divisor function, which is nontrivial for $(\log N)^{5+\varepsilon} \ll q \ll N^{1-\varepsilon}$. It was stated in [4] that under the Generalized Riemann Hypothesis,

one can get

$$\sum_{\substack{n \leq N \\ (n, q)=1}} \mu(n) e\left(\frac{a\bar{n}}{q}\right) \ll_{\varepsilon} q^{\frac{1}{2}} N^{\frac{1}{2}+\varepsilon}.$$

We also mention some progress on the relative topic. In 1998, E. Fouvry and P. Michel [6] proved that if q is a prime number, $g(x) = \frac{P(x)}{Q(x)}$ is any rational function with $P(x)$ and $Q(x)$ relatively prime monic polynomials in $\mathbb{Z}[x]$, then for $1 \leq N \leq q$, one has

$$\sum_{\substack{p \leq N \\ (Q(p), q)=1}} e\left(\frac{g(p)}{q}\right) \ll_{\varepsilon} q^{\frac{3}{16}+\varepsilon} N^{\frac{25}{32}},$$

where p runs through prime numbers, the implied constant also depends on the degrees of P and Q . This estimate is nontrivial for $N \leq q \ll N^{\frac{7}{6}-7\varepsilon}$. It was stated in [6] that the same method can produce

$$\sum_{\substack{n \leq N \\ (Q(n), q)=1}} \mu(n) e\left(\frac{g(n)}{q}\right) \ll_{\varepsilon} q^{\frac{3}{16}+\varepsilon} N^{\frac{25}{32}}$$

for the prime number q and $1 \leq N \leq q$. Some further results can be found in [5].

In 2011, E. Fouvry and I. E. Shparlinski [7] proved that for $(a, q) = 1$ and $N^{\frac{3}{4}} \leq q \leq N^{\frac{4}{3}}$, one has

$$\sum_{\substack{N < p \leq 2N \\ (p, q)=1}} e\left(\frac{a\bar{p}}{q}\right) \ll_{\varepsilon} q^{\varepsilon} (q^{\frac{1}{4}} N^{\frac{2}{3}} + N^{\frac{15}{16}}),$$

which is nontrivial for $N^{\frac{3}{4}} \leq q \ll N^{\frac{4}{3}-6\varepsilon}$. They also proved that if $(a, q) = 1$, then

$$\sum_{\substack{N < p \leq 2N \\ (p, q)=1}} e\left(\frac{a\bar{p}}{q}\right) \ll N \left(\tau^{\frac{1}{2}}(q) \frac{(\log N)^2}{q^{\frac{1}{2}}} + \tau(q) \frac{q^{\frac{1}{4}} (\log N)^{\frac{3}{2}}}{N^{\frac{1}{5}}} \right),$$

which is nontrivial for $(\log N)^{6+\varepsilon} \ll q \ll N^{\frac{4}{5}-\varepsilon}$. In 2012, R. C. Baker [1] gave improvement under some conditions.

When the first author visited the University of Montreal, Professor A. Granville suggested him to study the general sum

$$\sum_{\substack{n \leq N \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right), \tag{1.1}$$

where $f(n)$ is a multiplicative function satisfying $|f(n)| \leq 1$.

In this paper, we shall apply the method in Section 2 of [3], which is called as the finite version of Vinogradov's inequality, to give a nontrivial estimate for the sum in (1.1) when q is in a suitable range.

Theorem. Let $f(n)$ be a multiplicative function satisfying $|f(n)| \leq 1$, q ($\leq N^2$) be a positive integer and a be an integer with $(a, q) = 1$. Then we have

$$\sum_{\substack{n \leq N \\ (n, q) = 1}} f(n) e\left(\frac{a\bar{n}}{q}\right) \ll \sqrt{\frac{\tau(q)}{q}} N \log \log(6N) \quad (1.2)$$

$$+ q^{\frac{1}{4} + \frac{\varepsilon}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}}.$$

The estimate in (1.2) is nontrivial for

$$(\log \log(6N))^{2+\varepsilon} \ll q \ll N^{2-5\varepsilon}.$$

In a private communication, Ping Xi remarked that when q is a prime number, if Lemma 2 below is replaced by Theorem 16 in [2], then the upper bound in the above nontrivial range can be extended to $q \ll N^A$, where A is any given large constant.

Throughout this paper, we assume that N is sufficiently large and set

$$d_0 = \sqrt{\log \log(6N)}, \quad D_0 = e^{d_0} = \exp(\sqrt{\log \log(6N)}), \quad (1.3)$$

$$d_1 = d_0^2 = \log \log(6N), \quad D_1 = e^{d_1} = \log(6N).$$

Let p denote a prime number, $\tau(q)$ denote the divisor function, ε be a sufficiently small positive constant.

2. Some preliminaries

Write

$$S = \{n : 1 \leq n \leq N, n \text{ has a prime factor in } [D_0, D_1)\}, \quad (2.1)$$

$$T = \{n : 1 \leq n \leq N, n \text{ has no prime factor in } [D_0, D_1)\}.$$

Lemma 1. We have

$$|T| \ll \frac{N}{\sqrt{\log \log(6N)}}.$$

Proof. Let

$$P(N) = \prod_{D_0 \leq p < D_1} p.$$

We have

$$\begin{aligned} |T| &= \sum_{\substack{n \leq N \\ (n, P(N))=1}} 1 \\ &= \sum_{n \leq N} \sum_{d|(n, P(N))} \mu(d) \\ &= \sum_{d|P(N)} \mu(d) \sum_{\substack{n \leq N \\ d|n}} 1 \\ &= \sum_{d|P(N)} \mu(d) \left(\frac{N}{d} + O(1) \right) \\ &= N \sum_{d|P(N)} \frac{\mu(d)}{d} + O\left(2^{\pi(D_1)}\right) \\ &= N \prod_{D_0 \leq p < D_1} \left(1 - \frac{1}{p}\right) + O\left(2^{\frac{2D_1}{\log D_1}}\right) \\ &\ll N \frac{\log D_0}{\log D_1} + O\left(2^{\frac{2 \log(6N)}{\log \log(6N)}}\right) \\ &\ll \frac{N}{\sqrt{\log \log(6N)}}. \end{aligned}$$

Hence, Lemma 1 holds true.

By Lemma 1, we have

$$\sum_{\substack{n \leq N \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) = \sum_{\substack{n \leq N \\ n \in S \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right). \quad (2.2)$$

Let

$$P_r = \{p : e^r \leq p < e^{r+1}\}, \quad \text{if } [d_0] \leq r \leq [d_1]. \quad (2.3)$$

Then

$$\bigcup_{r=[d_0]+1}^{[d_1]-1} P_r \subseteq \{p : D_0 \leq p < D_1\} \subseteq \bigcup_{r=[d_0]}^{[d_1]} P_r.$$

The prime number theorem yields

$$|P_r| \ll \frac{e^r}{r}. \quad (2.4)$$

Write

$$S' = \{n : 1 \leq n \leq N, n \text{ has a prime factor in } \bigcup_{r=[d_0]}^{[d_1]} P_r\},$$

$$S'' = \{n : 1 \leq n \leq N, n \text{ has a prime factor in } \bigcup_{r=[d_0]+1}^{[d_1]-1} P_r\}.$$

Then

$$S'' \subseteq S \subseteq S'.$$

Hence,

$$\begin{aligned} |S \setminus S''| &\leq |S' \setminus S''| \ll \sum_{p \in P_{[d_0]}} \frac{N}{p} + \sum_{p \in P_{[d_1]}} \frac{N}{p} \\ &\ll N \left(\frac{|P_{[d_0]}|}{e^{[d_0]}} + \frac{|P_{[d_1]}|}{e^{[d_1]}} \right) \\ &\ll \frac{N}{d_0} = \frac{N}{\sqrt{\log \log(6N)}}. \end{aligned}$$

We note that

$$\begin{aligned} &|\{n : 1 \leq n \leq N, n \text{ has at least two prime factors in the} \\ &\quad \text{same one of } P'_r s \text{ } ([d_0] + 1 \leq r \leq [d_1] - 1)\}| \\ &\ll \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{p \in P_r} \sum_{p' \in P_r} \frac{N}{pp'} \\ &\ll N \sum_{r=[d_0]+1}^{[d_1]-1} \left(\frac{|P_r|}{e^r} \right)^2 \\ &\ll N \sum_{r=[d_0]+1}^{[d_1]-1} \frac{1}{r^2} \\ &\ll \frac{N}{d_0} = \frac{N}{\sqrt{\log \log(6N)}}. \end{aligned}$$

Therefore for

$$S''' = \{n : 1 \leq n \leq N, n \text{ has exact one prime factor} \\ \text{in one of } P'_r s \text{ } ([d_0] + 1 \leq r \leq [d_1] - 1)\},$$

we have

$$S''' \subseteq S''$$

and

$$|S'' \setminus S'''| \ll \frac{N}{\sqrt{\log \log(6N)}}.$$

The set S''' can be decomposed as

$$S''' = \bigcup_{r=[d_0]+1}^{[d_1]-1} S_r, \quad (2.5)$$

where

$$S_r = \{n : 1 \leq n \leq N, n \text{ has exact one prime factor in } P_r \quad (2.6)$$

and has no prime factor in $\bigcup_{i < r} P_i\}.$

By the prime number theorem, it is easy to see that each $S_r (r = [d_0] + 1, \dots, [d_1] - 1)$ is not empty. The sets S_r are disjoint from each other. Every element $n \in S_r$ can be written in exact one way as

$$n = py, \quad (2.7)$$

where $p \in P_r$, y has no prime factor in $\bigcup_{i \leq r} P_i$, $py \leq N$.

From the above discussion, we get

$$\begin{aligned} & \sum_{\substack{n \leq N \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) \\ &= \sum_{\substack{n \leq N \\ n \in S''' \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \\ &= \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{\substack{n \leq N \\ n \in S_r \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \\ &= \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{\substack{e^r \leq p < e^{r+1} \\ (p, q)=1}} \sum_{\substack{y \leq \frac{N}{p} \\ y \text{ has no prime factor in } \bigcup_{i \leq r} P_i \\ (y, q)=1}} f(py) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \quad (2.8) \\ & \quad + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{\substack{y \leq \frac{N}{e^r} \\ y \text{ has no prime factor in } \bigcup_{i \leq r} P_i \\ (y, q)=1}} f(y) \sum_{\substack{e^r \leq p < e^{r+1} \\ p \leq \frac{N}{y} \\ (p, q)=1}} f(p) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \\
&\quad + O\left(\frac{N}{\sqrt{\log \log(6N)}}\right) \\
&\ll \sum_{r=[d_0]+1}^{[d_1]-1} \sum_{\substack{y \leq \frac{N}{e^r} \\ (y, q)=1}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ p \leq \frac{N}{y} \\ (p, q)=1}} f(p) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \right| + \frac{N}{\sqrt{\log \log(6N)}}.
\end{aligned}$$

Let

$$Y = \frac{N}{e^r}. \quad (2.9)$$

We shall estimate the sum

$$\sum_1 = \sum_{\substack{y \leq Y \\ (y, q)=1}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ p \leq \frac{N}{y} \\ (p, q)=1}} f(p) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \right|. \quad (2.10)$$

Lemma 2. For the positive integer q and the integer b , we have

$$\sum_{\substack{X < n \leq Z \\ (n, q)=1}} e\left(\frac{b\bar{n}}{q}\right) \ll \left(\frac{Z-X}{q} + 1\right)(b, q) + q^{\frac{1}{2}+\varepsilon}. \quad (2.11)$$

Proof. Lemma 2.1 in [7] states that

$$\begin{aligned}
\sum_{\substack{X < n \leq Z \\ (n, q)=1}} e\left(\frac{b\bar{n}}{q}\right) &\ll \mu^2\left(\frac{q}{(b, q)}\right) \left(\frac{Z-X}{q} + 1\right) \cdot \frac{\varphi(q)}{\varphi\left(\frac{q}{(b, q)}\right)} \\
&\quad + \tau(q)\tau((b, q)) \log(2q)q^{\frac{1}{2}}.
\end{aligned}$$

Then the bounds

$$\begin{aligned}
\frac{\varphi(q)}{\varphi\left(\frac{q}{(b, q)}\right)} &= q \prod_{p|q} \left(1 - \frac{1}{p}\right) \cdot \left(\frac{q}{(b, q)} \prod_{p|\frac{q}{(b, q)}} \left(1 - \frac{1}{p}\right)\right)^{-1} \\
&= (b, q) \prod_{\substack{p|q \\ p \nmid \frac{q}{(b, q)}}} \left(1 - \frac{1}{p}\right) \leq (b, q)
\end{aligned}$$

and

$$\tau(q) \ll q^{\frac{\varepsilon}{4}}$$

produce the conclusion in Lemma 2.

3. The proof of Theorem

By Cauchy's inequality,

$$\sum_1 \leq Y^{\frac{1}{2}} \left(\sum_{\substack{y \leq Y \\ (y, q)=1}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ p \leq \frac{N}{y} \\ (p, q)=1}} f(p) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \right|^2 \right)^{\frac{1}{2}}. \quad (3.1)$$

An application of Lemma 2 to

$$\sum_2 = \sum_{\substack{y \leq Y \\ (y, q)=1}} \left| \sum_{\substack{e^r \leq p < e^{r+1} \\ p \leq \frac{N}{y} \\ (p, q)=1}} f(p) e\left(\frac{a\bar{p}\bar{y}}{q}\right) \right|^2$$

produces

$$\begin{aligned} \sum_2 &= \sum_{\substack{y \leq Y \\ (y, q)=1}} \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ p_1 \leq \frac{N}{y} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ p_2 \leq \frac{N}{y} \\ (p_2, q)=1}} f(p_1) \overline{f(p_2)} e\left(\frac{a(\bar{p}_1 - \bar{p}_2)\bar{y}}{q}\right) \\ &= \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1}} f(p_1) \overline{f(p_2)} \sum_{\substack{y \leq Y \\ p_1 \leq \frac{N}{y} \\ (y, q)=1}} e\left(\frac{a(\bar{p}_1 - \bar{p}_2)\bar{y}}{q}\right) \\ &\ll \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1}} \left| \sum_{\substack{y \leq Y \\ p_1 \leq \frac{N}{y} \\ (y, q)=1}} e\left(\frac{a(\bar{p}_1 - \bar{p}_2)\bar{y}}{q}\right) \right| \quad (3.2) \\ &\ll \sum_{\substack{e^r \leq p < e^{r+1} \\ (p, q)=1}} Y + \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1}} \left| \sum_{\substack{y \leq Y \\ p_1 \leq \frac{N}{y} \\ (y, q)=1}} e\left(\frac{a(\bar{p}_1 - \bar{p}_2)\bar{y}}{q}\right) \right| \\ &\ll Y e^r + \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1}} \left(\left(\frac{Y}{q} + 1 \right) (a(\bar{p}_1 - \bar{p}_2), q) + q^{\frac{1}{2} + \varepsilon} \right) \\ &\ll Y e^r + \left(\frac{Y}{q} + 1 \right) \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1}} (\bar{p}_1 - \bar{p}_2, q) + q^{\frac{1}{2} + \varepsilon} e^{2r}. \end{aligned}$$

We have

$$\begin{aligned}
& \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1}} (\bar{p}_1 - \bar{p}_2, q) \\
&= \sum_{k|q} k \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1 \\ (\bar{p}_1 - \bar{p}_2, q)=k}} 1 \\
&\leq \sum_{k|q} k \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1 \\ \bar{p}_2 \equiv \bar{p}_1 \pmod{k}}} 1 \\
&= \sum_{k|q} k \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1 \\ p_2 \equiv p_1 \pmod{k}}} 1.
\end{aligned} \tag{3.3}$$

In the above sum, if $k \geq e^{r+1}$, then $p_2 \equiv p_1 \pmod{k}$ and $p_1, p_2 < e^{r+1} \implies p_2 = p_1$, which contradicts the fact $p_2 \neq p_1$. Hence, it follows that

$$\begin{aligned}
& \sum_{k|q} k \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1 \\ p_2 \equiv p_1 \pmod{k}}} 1 \\
&= \sum_{\substack{k|q \\ k < e^{r+1}}} k \sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1 \\ p_2 \equiv p_1 \pmod{k}}} 1 \\
&\leq \sum_{\substack{k|q \\ k < e^{r+1}}} k \sum_{n_1 < e^{r+1}} \sum_{\substack{n_2 < e^{r+1} \\ n_2 \equiv n_1 \pmod{k}}} 1 \\
&\ll \sum_{\substack{k|q \\ k < e^{r+1}}} k \cdot e^{r+1} \cdot \frac{e^{r+1}}{k} \\
&\ll \tau(q) e^{2r}.
\end{aligned}$$

Thus we get the estimate

$$\sum_{\substack{e^r \leq p_1 < e^{r+1} \\ (p_1, q)=1}} \sum_{\substack{e^r \leq p_2 < e^{r+1} \\ (p_2, q)=1 \\ p_2 \neq p_1}} (\bar{p}_1 - \bar{p}_2, q) \ll \tau(q) e^{2r}. \tag{3.4}$$

By the above discussion, we have

$$\begin{aligned}\sum_2 &\ll Y e^r + \left(\frac{Y}{q} + 1\right) \tau(q) e^{2r} + q^{\frac{1}{2}+\varepsilon} e^{2r} \\ &\ll \frac{\tau(q)}{q} Y e^{2r} + Y e^r + q^{\frac{1}{2}+\varepsilon} e^{2r}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_1 &\ll Y^{\frac{1}{2}} \left(\frac{\tau(q)}{q} Y e^{2r} + Y e^r + q^{\frac{1}{2}+\varepsilon} e^{2r} \right)^{\frac{1}{2}} \\ &\ll \sqrt{\frac{\tau(q)}{q}} Y e^r + Y e^{\frac{r}{2}} + Y^{\frac{1}{2}} q^{\frac{1}{4}+\frac{\varepsilon}{2}} e^r \\ &\ll \sqrt{\frac{\tau(q)}{q}} N + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{4}+\frac{\varepsilon}{2}} N^{\frac{1}{2}} e^{\frac{r}{2}}.\end{aligned}$$

Applying this estimate to (2.8), we get

$$\begin{aligned}&\sum_{\substack{n \leq N \\ (n, q)=1}} f(n) e\left(\frac{a\bar{n}}{q}\right) \\ &\ll \sum_{r=[d_0]+1}^{[d_1]-1} \left(\sqrt{\frac{\tau(q)}{q}} N + \frac{N}{e^{\frac{r}{2}}} + q^{\frac{1}{4}+\frac{\varepsilon}{2}} N^{\frac{1}{2}} e^{\frac{r}{2}} \right) + \frac{N}{\sqrt{\log \log(6N)}} \\ &\ll \sqrt{\frac{\tau(q)}{q}} N \log \log(6N) + \frac{N}{\exp(\frac{1}{2} \sqrt{\log \log(6N)})} \\ &\quad + q^{\frac{1}{4}+\frac{\varepsilon}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}} \\ &\ll \sqrt{\frac{\tau(q)}{q}} N \log \log(6N) + q^{\frac{1}{4}+\frac{\varepsilon}{2}} N^{\frac{1}{2}} (\log(6N))^{\frac{1}{2}} + \frac{N}{\sqrt{\log \log(6N)}}.\end{aligned}$$

So far the proof of Theorem is complete.

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